1080 ft/sec).

6.7 TANGENTS AND NORMALS

We have already seen in the geometrical interpretation of the derivative of a curve $y = f(x)$ or $f(x, y) = 0$ that $\frac{dy}{dx}$ represents the slope of the tangent line to the curve at the point (x, y). In order to find an equation of the tangent to a given conic at some point on the conic, we shall first find the slope of the tangent at the given point by calculating $\frac{dy}{dx}$ from the equation of the conic at that point and then using the point – slope form of a line, it will be quite simple to write an equation of the tangent. Since the normal to a curve at a point on the curve is perpendicular to the tangent through the point of tangency, its equation can be easily written.

Example 1. Find equations of the tangent and normals to

(i) $$y^2 = 4ax \quad (1)$$

(ii) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (2)$$

(iii) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (3)$$

at the point (x_1, y_1).

Solution: (i). Differentiating (1) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a \quad \text{or} \quad \frac{dy}{dx} = \frac{2a}{y}$$

$$\left.\frac{dy}{dx}\right]_{(x_1, y_1)} = \frac{2a}{y_1} = \text{Slope of the tangent at } (x_1, y_1)$$

Equation of the tangent to (1) at (x_1, y_1) is

$$y - y_1 = \frac{2a}{y_1}(x - x_1) \quad \text{or} \quad yy_1 - y_1^2 = 2ax - 2ax_1 \quad \text{or} \quad yy_1 - 2ax = y_1^2 - 2ax_1$$

Adding $-2ax_1$ to both sides of the above equation, we obtain

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

Since (x_1, y_1) lies on $y^2 = 4ax$, so $y_1^2 - 4ax_1 = 0$

Thus equation of the required tangent is

$$yy_1 = 2a(x + x_1).$$

Slope of the normal $= \frac{-y_1}{2a}$ (negative reciprocal of slope of the tangent)

Equation of the normal is

$$y-y_1=\frac{-y_1}{2a}(x-x_1)$$

(ii) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Differentiating the above equation, w.r.t. x, we have

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \quad \text{or} \quad \frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y}$$

or $$\left.\frac{dy}{dx}\right]_{(x_1,y_1)}=\frac{-b^2}{a^2}\frac{x_1}{y_1}$$

Equation of the tangent to (2), at (x_1, y_1) is

$$y-y_1=\frac{-b^2}{a^2}\frac{x_1}{y_1}(x-x_1)$$

or $$\frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=\frac{-xx_1}{a^2}+\frac{x_1^2}{a^2} \quad \text{or} \quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

Since (x_1, y_1) lie on (2) so, $\frac{x^2}{a^2}+\frac{y_1^2}{b^2}=1$

Hence an equation of the tangent to (2) at (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Slope of the normal at (x_1, y_1) is $\frac{a^2y_1}{b^2x_1}$.

Equation of the normal at (x_1, y_2) is

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$$

or $b^2x_1y-b^2x_1y_1=a^2y_1x-a^2x_1y_1$ or $a^2y_1x-b^2x_1y=x_1y_1(a^2-b^2)$

Dividing both sides of the above equation by x_1y_1, we get

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$, as an equation of the normal.

(iii) Proceeding as in (ii), it is easy to see that equations of the tangent and normal to (3) at (x_1, y_1) are

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \text{ and } \frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2, \text{ respectively.}$$

Remarks

An equation of the **tangent** at the point (x_1, y_1) of any conic can be written by making replacements in the equation of the conic as under:

Replace x^2 by xx_1

y^2 by yy_1

x by $\frac{1}{2}(x+x_1)$

y by $\frac{1}{2}(y+y_1)$

Example 1. Write equations of the tangent and normal to the parabola $x^2 = 16y$ at the point whose abscissa is 8.

Solution: Since $x = 8$ lies on the parabola, substituting this value of x into the given equation, we find

$$64 = 16y \quad \text{or} \quad y = 4$$

Thus we have to find equations of tangent and normal at (8, 4).

Slope of the tangent to the parabola at (8, 4) is 1. An equation of the tangent the parabola at (8, 4) is

$$y-4 = x-8$$

or

$$x-y-4=0$$

Slope of the normal at (8, 4) is -1. Therefore, equation of the normal at the given point is

$$y-4=-(x-8)$$

or

$$x+y-12=0$$

Example 2. Write equations of the tangent and normal to the conic $\frac{x^2}{8}+\frac{y^2}{9}=1$ at the point $\left(\frac{8}{3},1\right)$.